The Zariski topology of Spec A (see Shaf V.1.3)

Recall that the closed sets in $\mathbb{A}^n$ are those which are the zeros of some radical ideal, I. The max'l ideals m corresponding to those points are those s.t. $I \subseteq m$. This gives rise to the following def:

Def: If R is a ring, and $E \subseteq R$ any set, define

$$V(E) \subseteq \operatorname{Spec} R$$

to be the prime ideals containing E. The topology on Spec$R$ in which the $V(E)$ are the closed sets is the Zariski topology. (Check this is actually a topology.)

The intersection of ideals is an ideal, so

$$V(E) = V(\underbrace{(E)}_{\text{ideal gen. by } E}).$$

Moreover, $\sqrt{I} = \bigcap \{\text{prime ideals containing } I\}$, so $V(I) = V(\sqrt{I})$.

A few easy-to-check facts:

1.) If $I, J \subseteq R$ are ideals, $V(IJ) = V(I) \cup V(J)$.

Pf: If $P \supseteq I$ or $P \supseteq J$, then $P \supseteq IJ$, so "$\supseteq$" holds.

If $P \supseteq IJ$ but doesn't contain I (WLOG) then there is some $a \in I$ s.t. $a \notin P$.

Thus, $\forall\ b \in J$, $ab \in P$, so $b \in P \Rightarrow J \subseteq P$.

$\Rightarrow$ "$\subseteq$" holds. $\square$

2.) $V(I) \subseteq V(J) \iff \sqrt{I} \supseteq \sqrt{J}$.

3.) $V(\sum_i I_i) = \bigcap_i V(I_i)$.

(Check 2.) and 3.))

The set of maximal ideals (maxSpec R) are exactly the set of closed points in Spec R.

Just as in the classical case, our "smallest" open sets (while still large!) are the complements of the closed set corresponding to a principal ideal. More precisely, if $f \in R$, define

$$D(f) := \operatorname{Spec} R \setminus V(f),$$

i.e. the prime ideals that do not contain f. The open sets of this form are called <u>distinguished open sets</u>.

Ex: Consider $f = xy \in k[x,y]$.

$$V(f) = \{ (x), (y), (x-\alpha, y), (x, y-\beta) \}$$

closed points

$D(f) = \{P \mid f \notin P\}$. In the language of affine varieties, $f \notin P \Leftrightarrow$ there is some point in the variety corresponding to P on which f doesn't vanish. i.e. some pt (a,b) w/ $a \neq 0, b \neq 0$, i.e. it doesn't lie in the coordinate axes.

<u>Ex</u>: If R has no zero divisors then (0) is prime. Its closure is then $V(0) = \operatorname{Spec} R$, the whole space. Points whose closure is the whole space are called <u>generic points</u>.

When is $P \in \operatorname{Spec} R$ a generic point?
When $P \subseteq \bigcap\{\text{all primes}\} = \mathfrak{N}(R)$, the <u>nilradical</u>.

$$\mathfrak{N}(R) = \{f \mid f^n = 0, \text{ some } n\}$$

Thus, $\operatorname{Spec} R$ has a generic point $\Leftrightarrow \mathfrak{N}(R)$ is prime.
$(\mathfrak{N}(R))$

<u>Ex</u>: $R = k[x,y]/(xy)$. Then $\mathfrak{N}(R) = (0)$, but (0) isn't prime, so there is no generic point of $\operatorname{Spec} R$.

However, notice that $\operatorname{Spec} R = V(x) \cup V(y)$

Set $S = k[x,y]/(x)$. The quotient $R \to S \cong R/(x)$ induces an inclusion $f: \operatorname{Spec} S \to \operatorname{Spec} R$, whose image is $V(x)$.

(In fact, any surjection of rings induces a closed immersion on Spec.)

So
$$\begin{array}{ccc} \operatorname{Spec} R = & V(x) & \cup\ V(y) \\ & \wr\wr & \wr\wr \\ & \operatorname{Spec}\left(k[x,y]/(x)\right) & \operatorname{Spec}\left(k[x,y]/(y)\right) \\ & \wr\wr & \wr\wr \\ & \operatorname{Spec} k[y] & \operatorname{Spec} k[x] \end{array}$$
both of which have (0) as generic point.

Ex: If $\mathcal{O}_x$ is the local ring of a regular point on a curve, then $\operatorname{Spec} \mathcal{O}_x = \{\underset{\text{generic point}}{(0)}, \underset{\text{closed point}}{m_x}\}$

Caution: The topology on Spec R is just one piece of data of the "scheme."

e.g. $\operatorname{Spec} \mathbb{C}$, $\operatorname{Spec} \mathbb{Z}/4\mathbb{Z}$, and $\operatorname{Spec} \mathbb{Q}[x]/(x^2)$

are all one point sets and thus homeomorphic.

However, they are all very different as schemes, as we will see, since the original ring is an important part of the data.

However, not every scheme is of the form Spec R, so we need sheaves to describe schemes more generally.